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NON-ABELIAN SHARP PERMUTATION p-GROUPS

BY

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ABSTRACT

A permutation group G of finite degree d is called a sharp permutation group of type $\{k\}$, k a non-negative integer, if every non-identity element of G has k fixed points and $|G| = d-k$. We characterize sharp non-abelian p-groups of type $\{k\}$ for all k.

1. Introduction

Let G be a permutation group of finite degree d and let k be a non-negative integer. G is called a permutation group of type $\{k\}$ if every non-identity element of G has k fixed points; a group of type $\{k\}$ for some positive integer k is called a group of finite type. G is called a **sharp permutation group of type** $\{k\}$ if it is a group of type $\{k\}$ and $|G| = d - k$ (see [3]). From the Orbit Counting Lemma ([3, Theorem 2.2]) it follows that a permutation group of type $\{k\}$ is sharp if and only if it has exactly $k + 1$ orbits. For example, each non-trivial permutation group with k global fixed points and one regular orbit is a sharp permutation group of type $\{k\}$. Moreover, a sharp permutation group of type $\{k\}$ which has $h \leq k$ global fixed points is isomorphic to a sharp permutation group of type $\{k - h\}$. Therefore, to avoid trivialities we consider only sharp permutation groups without global fixed points and regular orbits and we call them sharp **irredundant** (permutation) groups of type $\{k\}$. Note that the absence of regular orbits forces k to be positive.

Finite groups admitting a faithful representation as sharp irredundant permutation groups of finite type have been investigated in [4] and [5]. A complete

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description is given, except for the case when G is a non-abelian p-group, in which case only sharp irredundant non-abelian p-groups of type $\{p\}$ are characterized. In this paper we complete the classification of sharp irredundant non-abelian p-groups of type $\{k\}$ for all k. We prove the following result.

THEOREM 1: *Let G be a finite non-abelian p-group. Then G has a faithful representation as a sharp irredundant group of type $\{k\}$, for some k such that $p^n \leq k < p^{n+1}$, if and only if it has the following properties:*

(i) *G/G' is an elementary abelian p-group of order p^{2n};*

(ii) *there exists a subgroup $N \geq G'$ of index p^n such that for each $g \in G \setminus N$, g has order p and $|C_N(g)| = p^n$;*

(iii) *for each $g \in G \setminus N$ there exists a complement A_g for N in G such that $g \in A_g$ and:*

  (a) *the elements of the set $\{A_g \mid g \in G\}$ split into p^n conjugacy classes;*

  (b) *the set $\pi = \{A_g G'/G' \mid g \in G \setminus N\} \cup \{N/G'\}$ is a partition of G/G'.*

Recall that a **partition** of a group G is a set π of non-trivial subgroups of G such that each non-identity element of G belongs to exactly one of them. The elements of π are called components. π is **non-trivial** if each component is a proper subgroup. π is **normal** if $X^g \in \pi$ for each $X \in \pi$, $g \in G$. For basic concepts and results on groups with partition we refer to [8].

By Theorem 1 in [4] a necessary condition for a finite group to have a faithful representation as a sharp irredundant group of finite type is to have a non-trivial normal partition. Therefore, all groups considered in this paper are groups with a non-trivial normal partition. Recall that a finite p-group G has a non-trivial normal partition if and only if it has a proper normal subgroup N such that every element outside N has order p (see [8]). When G is a finite non-abelian 2-group, such an N, when it exists, has index 2 in G. Thus Theorem 8 in [4] implies that a finite non-abelian 2-group has a faithful representation as a sharp irredundant group of type $\{k\}$ if and only if it is a dihedral 2-group and $k = 2$.

Further information on the structure of sharp irredundant non-abelian p-groups of finite type, $p > 2$, are collected in the following theorem.

THEOREM 2: *Let G be a sharp irredundant non-abelian p-group of finite type $\{k\}$ on a set Ω, $p > 2$. Let $|G : G'| = p^{2n}$ and let c be the nilpotency class of G. Then:*

(i) *$|\gamma_i(G) : \gamma_{i+1}(G)| = p^n$ for all $i = 2, \ldots, c-1$ and $|\gamma_c(G)| \leq p^n$;*

(ii) *the lower central series and the upper central series of G coincide;*

(iii) *for every normal subgroup H of G there exists an index i such that $\gamma_{i+1}(G) \leq H \leq \gamma_i(G)$;*

(iv) *for every normal subgroup H of G such that $H \leq G'$, the quotient group G/H acts on the set of the H-orbits of Ω as an irredundant sharp group of type $\{k\}$.*

Note that condition (iii) in Theorem 2 says that G is a normally constrained p-group (see [2]).

About possible values for k, it is immediate to see that when G is a sharp p-group of type $\{k\}$, then p divides k. When G is non-abelian, if $p = 2$, as we observed above, $k = 2$; if p is odd, possible values for k are given by Theorem 2.

THEOREM 3: *Let G be a finite non-abelian p-group, $p > 2$, and suppose that G has a faithful representation as sharp irredundant group of type $\{k\}$. Let $|G : G'| = p^{2n}$. Then $k = p^n + |\pi| - 1$, for some partition π of an elementary abelian p-group of order p^n. Moreover, for each partition π of an elementary abelian p-group of order p^n, G has a faithful representation as a sharp irredundant group of type $\{p^n + |\pi| - 1\}$.*

In Section 5 we construct some examples of irredundant sharp non-abelian p-groups of finite type. Moreover, we determine all non-abelian p-groups of order at most p^5, $p > 2$, with a faithful representation as an irredundant sharp permutation group of finite type. Furthermore, we show that the values of $k < p^5$ for which there exists an irredundant sharp non-abelian p-group of type $\{k\}$, $p > 2$, are: $p, p^2, p^2 + p, p^3, p^3 + p^2, p^3 + p^2 + p, p^4$ and $p^4 + p^3 + p^2 - (m-1)p$, for $1 \leq m \leq p^2 + 1$.

NOTATION. Let G be a group. If $\rho_i, i = 0, \dots, r$ are permutation representations of G on some sets Ω_i, we denote by $\sum_{i=0}^r \rho_i$ the permutation representation ρ of G on the disjoint union $\Omega = \dot{\bigcup}_{i=0}^r \Omega_i$ defined by the position $\omega g^\rho = \omega g^{\rho_i}$ if $\omega \in \Omega_i$, $g \in G$. Moreover, we denote by ρ_X the standard permutation representation of G on the right cosets of the subgroup X.

2. Some general properties

Let G be an irredundant group of type $\{k\}$ on a set Ω. By Theorem 1 in [4] the non-trivial stabilizers of the subsets of Ω of size k form a non-trivial normal partition of G denoted by π_Ω. If G is a p-group, then $Z(G) \neq 1$ and so, by Lemma 3.5.4 in [8], any non-trivial normal partition of G contains at least one component that is a normal subgroup of G.

LEMMA 4: *Let G be a group with a non-trivial partition π such that every component of π is a normal subgroup of G. Then G is abelian.*

Proof: If X and Y are distinct components of π, then trivially $[X,Y] \leq X \cap Y = 1$. Therefore, we need only show that each component is abelian. So let $X \in \pi$, $x, y \in X$ and choose $z \notin X$. Then $zx \notin X$ and so $[z,y] = 1 = [zx,y]$, thus $[zx,y] = [x,y] = 1$ as claimed. ∎

PROPOSITION 5: *Let G be a finite group acting faithfully as an irredundant group of type $\{k\}$ on a set Ω. Let N be a normal component of π_Ω and let Z be a central subgroup of G contained in N. Then G/Z acts faithfully as a group of type $\{k\}$ on the set Ω/Z of the Z-orbits of Ω. Moreover:*

(i) *if $Z \neq N$, then the action of G/Z on Ω/Z is irredundant;*

(ii) *if G is sharp on Ω and G is non-abelian, then G/Z is sharp with no regular orbits on Ω/Z;*

(iii) *if G is sharp on Ω and either Z is properly contained in N or $Z \leq G'$, then G/Z acts on Ω/Z as a sharp irredundant group of type $\{k\}$.*

Proof: First of all note that any stabilizer in G of a point of Ω either contains N (and so Z), or intersects N (and so Z) trivially. Hence Z acts on each G-orbit either trivially or semi-regularly. Moreover, by definition, N is the intersection of the stabilizers of the points that are trivial Z-orbits.

Let us prove that Z is the kernel of the action of G on Ω/Z. Clearly Z is contained in the kernel. Conversely, let $x \in G$ act trivially on Ω/Z. Then, in particular, x fixes all the points of Ω that are trivial Z-orbits; thus $x \in N$. On the other hand, x leaves invariant all the regular Z-orbits. Thus if ω^Z is a regular Z-orbit, then $\omega x = \omega z$ for some $z \in Z$, that is $xz^{-1} \in \mathrm{St}_G(\omega)$. Thus $xz^{-1} \in \mathrm{St}_G(\omega) \cap N = 1$ and $x \in Z$. Hence G/Z acts faithfully on Ω/Z as claimed.

Let us prove now that every non-identity element of G/Z has k fixed points on Ω/Z. Let $gZ \in G/Z$, $g \notin Z$, let Γ be the set of points of Ω that are trivial Z-orbits fixed by g and set $|\Gamma| = h$. We need to show that the number of regular Z-orbits fixed by g is exactly $k - h$. A regular Z-orbit ω^Z is fixed by gZ if and only if $\omega g = \omega z$ for some $z \in Z$, that is ω is fixed by some element t of gZ. In such a case, t acts trivially on ω^Z. Moreover, since ω^Z is a regular Z-orbit, ω cannot be fixed by two distinct elements of gZ. Thus distinct elements of gZ can act trivially only on distinct regular Z-orbits. Since any element of gZ has k fixed points in Ω and acts trivially on Γ, it acts trivially on $(k-h)/|Z|$ regular Z-orbits. Hence the number of Z-orbits that are fixed by gZ is

$$h + \sum_{z \in Z} \frac{k-h}{|Z|} = h + (k-h) = k,$$

as claimed.

Let us study now regular orbits and global fixed points in Ω/Z. Suppose that $\Delta/Z \subseteq \Omega/Z$ is a G/Z-regular orbit. Then, since G does not have regular orbits on Ω and Z acts on each G-orbit either trivially or semi-regularly, Z must act trivially on Δ. Thus $|\Delta| = |\Delta/Z| = |G/Z|$. Hence Z is the stabilizer of every point of Δ. In particular, $Z = N$. On the other hand, suppose that $\omega^Z \in \Omega/Z$ is a global fixed point for G/Z. Then, since G does not have global fixed points in Ω, ω^Z is a regular Z-orbit in Ω and so $\mathrm{St}_G(\omega) \cap Z = 1$. Moreover, $\mathrm{St}_G(\omega)$ intersects non-trivially every coset of Z. Thus $\mathrm{St}_G(\omega)Z = G$ and, since Z is a central subgroup of G, $G' \leq \mathrm{St}_G(\omega)$.

By the previous discussion about global fixed points and regular orbits in Ω/Z, (i) follows at once, since if $Z \neq N$, then no stabilizer of a point of Ω is a complement for Z in G.

To prove (ii) note that if G is sharp, then G/Z is sharp on Ω/Z as well, since G/Z is of type $\{k\}$ and the number of orbits did not change. Thus if $\Delta/Z \subseteq \Omega/Z$ is a regular orbit for G/Z, then since every non-identity element of G/Z moves exactly $|G/Z|$ points, all the points outside Δ/Z are global fixed points. Hence by what we have seen above, $Z = N$ is a point-stabilizer and every point-stabilizer distinct from Z is a normal subgroup of G. So every component of the partition π_Ω is normal. Hence, by Lemma 4, G is abelian: a contradiction. This proves (ii).

To prove (iii), note that if $Z < N$ the result follows from (i) and from the fact that G/Z is sharp on Ω/Z, provided G is sharp on Ω. If $Z \leq G'$, then we may assume $G' \neq 1$, otherwise $Z = 1$ and there is nothing to prove. Hence by (ii), G/Z has no regular orbit. If it has a global fixed point ω^Z, then $G' \leq \mathrm{St}_G(\omega)$ and $Z = Z \cap \mathrm{St}_G(\omega) = 1$: a contradiction. So (iii) holds. ∎

Remark: Note that in the hypothesis of Proposition 5(iii), when Z is a proper subgroup of N we have that N/Z is a normal component of $\pi_{\Omega/Z}$. In fact, let $\Delta \subseteq \Omega$ be the set of the k points fixed by N. Since $Z \leq N$, every point of Δ is a trivial Z-orbit and so Δ/Z is a subset of Ω/Z of size k. The stabilizer of Δ/Z is N/Z.

LEMMA 6: *Let G be an irredundant group of type $\{k\}$ on a set Ω and let N be a component of π_Ω. Then $|C_N(g)| \leq k$ for each $g \notin N$.*

Proof: Let $g \notin N$. Then there exists $\omega \in \Omega$ such that $\mathrm{St}_G(\omega) \cap N = 1$ and $g \in \mathrm{St}_G(\omega)$. Clearly $\omega t_1 \neq \omega t_2$ for every $t_1, t_2 \in C_N(g)$, $t_1 \neq t_2$, and $\omega tg = \omega gt = \omega t$

for every $t \in C_N(g)$. Since g has k fixed points in Ω, it follows that $|C_N(g)| \leq k$. ∎

3. Extra-special p-groups

In this section we consider the particular case of extra-special p-groups and p-groups with derived subgroup of order p, for odd primes p. It is well known that an extra-special p-group of exponent greater than p does not have a non-trivial partition. Thus it cannot be faithfully represented as a sharp irredundant group of finite type.

THEOREM 7: *Let G be an extra-special p-group of order p^{2n+1}, $n \geq 1$, $p > 2$, and exponent p. Then G can be faithfully represented as a sharp irredundant group of type $\{k\}$ if and only if $k = p^n + h$, where either $h = 0$ or $\{h\}$ is the type of a sharp irredundant elementary abelian p-group of order p^n.*

Moreover, if G acts on a set Ω as a sharp irredundant group of type $\{p^n + h\}$, then π_Ω contains a unique normal component N which is an abelian group of order p^{n+1}, every point-stabilizer not containing N is an abelian complement of N and every point-stabilizer containing N is either N itself when $h = 0$ or the preimage, via the natural epimorphism of G onto G/N, of a point-stabilizer of a faithful representation of G/N as a sharp irredundant group of type $\{h\}$.

Proof: Let G be an extra-special p-group of exponent p and order p^{2n+1}, $n \geq 1$, $p > 2$. The elementary abelian p-group $V = G/Z(G)$ may be regarded as a symplectic space of dimension $2n$ over the field with p elements $\mathbb{F}_p$ with respect to the form $f : V \times V \to \mathbb{F}_p$, $(xZ(G), yZ(G)) \mapsto [x, y]$. It is well known that there is an $\mathbb{F}_p$-isomorphism of the symplectic space W of dimension 2 over the field with p^n elements onto V, mapping totally isotropic subspaces into totally isotropic subspaces. Thus the set of images of one-dimensional subspaces of W is a partition π of V consisting of maximal totally isotropic subspaces. Let $A_0 = N, A_1, \ldots, A_{p^n}$ be the full preimages of the elements of π with respect to the natural projection of G onto V. Then any A_i is an elementary abelian p-group and $A_i \cap A_j = Z(G)$ for each $i \neq j$. For $i = 1, \ldots, p^n$, let X_i be a complement of $Z(G)$ in A_i. Clearly $N_G(X_i) = A_i$, so that X_i has p^n conjugates. Moreover, every conjugate of X_i is a maximal subgroup of A_i not containing $Z(G)$. Since A_i has $p^n + p^{n-1} + \cdots + 1$ maximal subgroups and those containing $Z(G)$ are $p^{n-1} + p^{n-2} + \cdots + 1$ in number, it follows that the conjugates of X_i are exactly the maximal subgroups of A_i not containing $Z(G)$. Thus any element of $A_i \setminus Z(G)$ is contained in p^{n-1} conjugates of X_i. Clearly $X_i^g \cap X_j^h = 1$ for all

$g, h \in G$, $i \neq j$. Hence if $g \in A_i \setminus Z(G)$, $i \neq 0$, then $g^{\rho_{X_i}}$ has p^n fixed points, while $g^{\rho_{X_j}}$, $j \neq i$, and g^{ρ_N} act fixed-point-freely. If $1 \neq g \in N$, then g^{ρ_N} has p^n fixed points and $g^{\rho_{X_i}}$ acts fixed-point-freely for every $i = 1, \dots, p^n$. Hence $\rho = \rho_N + \sum_{i=1}^{p^n} \rho_{X_i}$ is a faithful representation of G as a sharp irredundant group of type $\{p^n\}$.

If $\sigma = \sum_{i=1}^{m} \rho_{M_i/N}$ is a faithful representation of G/N as a sharp irredundant group of type $\{h\}$, then $\rho = \sum_{i=1}^{m} \rho_{M_i} + \sum_{i=1}^{p^n} \rho_{X_i}$ is a faithful representation of G of type $\{p^n + h\}$. In fact, if $1 \neq g \in N$, then $g^{\rho_{M_i}}$ is the identity for every $i = 1, \dots, m$ and $g^{\rho_{X_i}}$ acts fixed-point-freely. Hence g^{ρ} has as many fixed points as the degree of the representation σ, that is $|G/N| + h = p^n + h$. If $g \notin N$ then $g \in M_j$ if and only if $gN \in M_j/N$. Thus, since M_j is a normal subgroup of G, $g^{\rho_{M_j}}$ has $|G : M_j|$ fixed points if $gN \in M_j/N$ and none otherwise. Therefore $g^{\sum_{j=1}^{m} \rho_{M_j}}$ has as many fixed points as the number of fixed points of $(gN)^{\sigma}$, that is h. Hence g^{ρ} has $p^n + h$ fixed points, as claimed.

Now let ρ be a faithful representation of G as a sharp irredundant group of type $\{k\}$ on a set Ω. Since $|G'| = p$, G' acts on each orbit either trivially or semi-regularly. Let $\mathcal{O}$ be an orbit on which G' acts semi-regularly and let X be the point-stabilizer of a point in $\mathcal{O}$. By Proposition 5(iii), G/G' acts on Ω/G' as a sharp irredundant group of type $\{k\}$ and so, by the result in [5], the lengths of the G/G'-orbits on Ω/G' are the orders of the components of a non-trivial partition of G/G'. Since $|G/G'| = p^{2n}$, the maximum of such orders is p^n. Clearly, $\mathcal{O}/G'$ is a G/G'-orbit in Ω/G' of length $|\mathcal{O}|/p$. Thus $|\mathcal{O}|/p \leq p^n$ and $|\mathcal{O}| \leq p^{n+1}$, that is $|X| \geq p^n$. On the other hand, X is an abelian subgroup of G not containing G' and so it has order at most p^n. Hence $|X| = p^n$, that is every orbit in Ω on which G' acts semi-regularly has length p^{n+1}. Since every non-identity element of G moves exactly $|G|$ points in Ω, if a denotes the number of orbits on which G' acts semi-regularly, we have that $ap^{n+1} = |G|$, and so $a = p^n$.

Now let N be the normal component of π_Ω containing G'. Since $G' = Z(G)$ is contained in every non-trivial normal subgroup of G, it is clear that N is the unique normal component of π_Ω. Moreover, since by definition of N, $N \cap X = 1$ for each point-stabilizer X not containing G', we have that $|N| \leq p^{n+1}$. On the other hand, every element outside N belongs to some point-stabilizer not containing G'. Moreover, if X is a point-stabilizer not containing G', then every conjugate of X is contained in XG'. Therefore we must have $|G \backslash N| \leq |XG' \backslash G'|a$, that is $p^{2n+1} - |N| \leq (p^{n+1} - p)p^n$, whence $|N| \geq p^{n+1}$. Thus $|N| = p^{n+1}$.

Now we want to show that every element outside G' fixes exactly p^n points in those orbits on which G' acts semi-regularly. So let X be a point-stabilizer not containing G'. Then XG' is an abelian subgroup of order p^{n+1}. We claim that $XG' = N_G(X)$. In fact if $N_G(X) > XG'$, then there exists a subgroup H such that $X < H \le N_G(X)$ and $|H : X| = p^2$. Hence $H' \le X \cap G' = 1$ and H is abelian: a contradiction since G does not contain abelian subgroups of order p^{n+2}. Thus $XG' = N_G(X)$ and X has p^n conjugates. The argument used in the first part of the proof for detecting the conjugates of X_i applies here and shows that $XG' \setminus G' = \bigcup_{g \in G} X^g \setminus \{1\}$. Then by a counting argument it follows that a non-identity element of X is contained in p^{n-1} distinct conjugates of X and cannot be contained in any point-stabilizer not containing G' and non-conjugate with X. Hence a non-identity element of X fixes exactly p^n points in those orbits on which G' acts semi-regularly. Thus if N is a point-stabilizer, then $k = p^n$ and we are done. If N is not a point-stabilizer, let $S_1, \ldots, S_t$ be the point-stabilizers containing N. If X is a point-stabilizer not containing G', then $\tau = \sum_{i=1}^{t} \rho_{X \cap S_i}$ is a faithful representation of X as an irredundant group of type $\{k - p^n\}$ and the number of orbits is $t = k + 1 - p^n$. Since $XN = G$ the thesis follows. ∎

Remark: Let A be an elementary abelian p-group of order p^n. Then by [5] every faithful representation of A as a sharp irredundant group of type $\{h\}$ is such that there exists a partition π of A with $h = |\pi| - 1$. Since the maximum number of components of a partition of A is $(p^n - 1)/(p - 1)$, we have that h is at most $p^{n-1} + p^{n-2} + \cdots + p$. Therefore from Theorem 7 it follows that if G is a sharp irredundant extra-special p-group of order p^{2n+1} and exponent $p > 2$ of type $\{k\}$, then $p^n \le k \le p^n + p^{n-1} + \cdots + p < p^{n+1}$.

PROPOSITION 8: *Let p be an odd prime and let G be a finite p-group with $|G'| = p$. If G has a faithful representation as a sharp irredundant group of finite type, then G is an extra-special p-group of exponent p.*

Proof: Let G be a finite p-group with $|G'| = p$ and let us assume that G has a faithful representation as a sharp irredundant group of finite type $\{k\}$ on a set Ω. Then G has a non-trivial partition and so it is generated by elements of order p. Since it has class 2, it has exponent p. Therefore, in order to prove that G is an extra-special p-group of exponent p it is enough to show that $Z(G) = G'$. In order to obtain a contradiction, suppose that $Z(G) > G'$ and choose G of minimal order with this property.

Let Z be a central subgroup of order p, $Z \neq G'$. Then the component of π_Ω containing Z is normal in G and so, by Proposition 5(ii), G/Z acts faithfully as a sharp group of type $\{k\}$ on the set of all Z-orbits Ω/Z, without regular orbits. Hence it has a faithful representation as a sharp irredundant group of type $\{h\}$, with $0 < h \leq k$. Since $|(G/Z)'| = p$, the minimality of $|G|$ implies that G/Z is an extra-special p-group. Set $|G/Z| = p^{2n+1}$, whence $|G| = p^{2n+2}$. Then, by Theorem 7, Ω/Z contains at most $p^n + 1$ G/Z-orbits of length greater than or equal to p^n. Clearly, if X is a point-stabilizer not containing G', then X is abelian. So XZ/Z is an abelian subgroup of the extra-special p-group G/Z, and thus must satisfy $|XZ/Z| \leq p^{n+1}$ and $|XZ/Z| \leq p^n$ if further $G'Z/Z \not\leq XZ/Z$. Now if $X \cap Z = 1$, then clearly we have that $|X| \leq p^{n+1}$. If $X \geq Z$, then $G'Z/Z \not\leq XZ/Z$, since $G' \not\leq X$. Thus in both cases, $|X| \leq p^{n+1}$. Hence the orbits on which G' acts semi-regularly have length at least p^{n+1}.

By Proposition 5, G/G' acts on Ω/G' as a sharp irredundant group of type $\{k\}$. By the result in [5] the lengths of the orbits of a sharp irredundant abelian group correspond to the sizes of the components of a non-trivial partition of the group itself. Therefore, since $|G/G'| = p^{2n+1}$, we have that G/G' may have at most only one orbit on Ω/G' of length greater than p^{n+1}. Therefore, among the G-orbits on which G' acts semi-regularly, at most only one of them has length p^{n+2} and all the others have length p^{n+1}.

Let a be the number of orbits of length p^{n+1} on which G' acts semi-regularly. Then, since every non-identity element of G moves exactly $|G|$ points, by counting the points moved by a non-identity element of G', we have that $ap^{n+1} + p^{n+2} \geq p^{2n+2}$, whence $a \geq p^{n+1} - p$. Thus we have that G/Z acts on Ω/Z with at least $p^{n+1} - p$ orbits of length at least p^n: a contradiction. ∎

4. Proofs of the main results

In this section we prove Theorem 1, Theorem 2 and Theorem 3.

Proof of Theorem 1: Let G be a finite non-abelian p-group. First we prove that if G satisfies conditions (i)–(iii) in Theorem 1, then G has a faithful representation as a sharp irredundant group of type $\{p^n\}$. Namely, we claim that if $\mathcal{X}$ is a set of representatives for the conjugacy classes of the subgroups A_g, $g \in G \setminus N$, defined in (iii), then $\rho = \rho_N + \sum_{X \in \mathcal{X}} \rho_X$ is a faithful representation of G as a sharp irredundant group of type $\{p^n\}$.

It is immediate that ρ is faithful and irredundant. Moreover, $|\mathcal{X}| = p^n$ by condition (iii.a); so the number of orbits of ρ is $p^n + 1$. Hence we need only prove that ρ is of type $\{p^n\}$. First note that if $X, Y \in \mathcal{X}$ and $X \neq Y$, then $X \cap Y^a = 1$ for all $a \in G$. In fact, if we suppose that $X \cap Y^a \neq 1$ for some $a \in G$, then $XG' \cap YG' > G'$ and so $XG' = YG'$, since by (iii.b), XG'/G' and YG'/G' are components of the partition π of G/G'. Then π has at most p^n components, each one of order p^n: a contradiction since G/G' has order p^{2n}.

Now let $g \in G \setminus N$ and let $X \in \mathcal{X}$ be such that g belongs to a conjugate of X. Actually, there is no loss of generality in assuming $g \in X$. By the previous observation g^ρ acts fixed-point-freely on the right cosets of N and of Y, for $X \neq Y \in \mathcal{X}$. Therefore, the number of fixed points of g^ρ is equal to the number of the elements $a \in N$ such that $Xag = Xa$, or equivalently $g \in X \cap X^a$. Now $g \in X \cap X^a$ if and only if $[g, a] \in X^a \cap G' = 1$, that is $a \in C_N(g)$. Thus the number of fixed points of g^ρ is $|C_N(g)| = p^n$, by (ii). Trivially, if $1 \neq g \in N$, then g^ρ acts trivially on the right cosets of N and with no fixed points on the right cosets of X, for $X \in \mathcal{X}$. Therefore ρ is of type $\{p^n\}$, as desired.

Now suppose that G has a faithful representation as a sharp irredundant group of type $\{k\}$ on a set Ω, with $p^n \leq k < p^{n+1}$. If $p = 2$, then G has exponent greater than 2 since it is non-abelian and thus, by a result of Hughes [6], $|G : H_2(G)| = 2$, where $H_2(G)$ is the Hughes subgroup of G (that is the subgroup generated by elements of order greater than 2). Then the result follows from Theorem 8 in [4]. So for the remainder of the proof we assume $p > 2$. We prove that π_Ω contains a unique normal component and G satisfies conditions (i)–(iii), where N is the normal component of π_Ω and the set of the complements A_g for N in G is the set of all point-stabilizers not containing N. If $|G'| = p$, then by Proposition 8, G is an extra-special p-group and so by Theorem 7 we can conclude. In particular, we are done when $|G| = p^3$. Hence we may assume that $|G| > p^3$, $|G'| > p$ and that the claim is true for groups of order smaller than $|G|$. The proof is divided into many steps.

(a) *G/G' is an elementary abelian p-group of order p^{2n}.*

Let M be a maximal subgroup of G', normal in G. Then repeated applications of Proposition 5(iii) yield that G/M is a sharp irredundant group of type $\{k\}$, with derived subgroup of order p. So by Proposition 8 it is an extra-special p-group. Hence from Theorem 7 it follows that $(G/M)/(G'/M)$ is an elementary abelian p-group of order p^{2n} and so the same holds for G/G'.

(b) *π_Ω contains a unique normal component $N \geq G'$ such that $|G : N| = p^n$ and every point-stabilizer not containing N is a complement for N in G.*

Let Z be a subgroup of order p of $G' \cap Z(G)$ and let N be a component of π_Ω containing Z. Then N is a normal subgroup of G. Set $G/Z = \overline{G}$ and for each subgroup R of G denote $\overline{R} = RZ/Z$. By Proposition 5(iii), $\overline{G}$ is a sharp irredundant p-group of type $\{k\}$ on the set $\Gamma = \Omega/Z$ of Z-orbits of Ω. Moreover, since $|G'| > p$, $\overline{G}$ is non-abelian. Then by the inductive hypothesis the claim holds for $\overline{G}$. Let $\overline{M}$ be the unique normal component of π_Γ. If $\overline{N} \neq 1$, then by the Remark after Proposition 5, $\overline{N}$ is a normal component of π_Γ. Thus the uniqueness of $\overline{M}$ implies that either $\overline{N} = 1$ or $\overline{M} = \overline{N}$. If $\overline{N} = 1$, then there exists a point-stabilizer H in G containing N such that $\overline{H} \cap \overline{M} = 1$. Since the claim holds for $\overline{G}$, $\overline{H}$ is abelian of order p^n. On the other hand, since any non-identity element of N fixes by right multiplication all the cosets of H, $|G : H| \leq k < p^{n+1}$. Thus $|\overline{G}| \leq p^{2n}$. It follows that $\overline{G}$ is abelian since $|G : G'| = p^{2n}$ by (a). A contradiction. Therefore $\overline{M} = \overline{N}$ and so N is the unique normal component of π_Ω. Moreover, $|G : N| = |\overline{G} : \overline{N}| = p^n$. If X is a point-stabilizer not containing N, then $\overline{X}$ is a point-stabilizer in $\overline{G}$ not containing $\overline{N}$ and so it is a complement for $\overline{N}$ in $\overline{G}$. Hence X is a complement for N in G, as claimed.

(c) *For each $g \in G \setminus N$, $|C_N(g)| = p^n$.*

Let $\overline{G}$ be as in the proof of (b). Let $g \in G \setminus N$. By Lemma 6, $|C_N(g)| \leq k$ and so $|C_N(g)| \leq p^n$. On the other hand, $p^n = |C_{\overline{N}}(\overline{g})|$ by the inductive hypothesis, and by a result of Khukhro ([7, Theorem 1.6.1]) $|C_{\overline{N}}(\overline{g})| \leq |C_N(g)|$. Hence $|C_N(g)| = p^n$.

(d) *G has a faithful representation as an irredundant sharp group of type $\{p^n\}$.*

If $k = p^n$ there is nothing to prove. So suppose that $k \neq p^n$. To produce a faithful representation of G as irredundant sharp group of type $\{p^n\}$ we want to replace the orbits on which N acts trivially with a single orbit on which G acts as on the right cosets of N by right multiplication. It is trivial that this procedure produce a faithful representation of G with $p^n + 1$ orbits, in which every non-identity element of N has p^n fixed points. Thus, we need only prove that in the new representation also the elements outside N have p^n fixed points. So let $g \notin N$ and let X be a point-stabilizer containing g and not containing N. By (c), $|C_N(g)| = p^n$ and so g fixes at least p^n points in the orbit ω^G, where $\mathrm{St}_G(\omega) = X$. Since $X \cap N = 1$, ω^G is a regular N-orbit. To complete the proof we show that g fixes $k - p^n$ points in those orbits of Ω on which N acts trivially. Let M be a maximal subgroup of G', normal in G. Then as in the proof of (a), G/M is an extra-special p-group of order p^{2n+1} acting as a sharp irredundant group of type $\{k\}$ on the set Ω/M of all M-orbits of Ω. By Theorem 7, $gM \in G/M \setminus N/M$ fixes $k - p^n$ points in those G/M-orbits on which N/M acts trivially. So g fixes $k - p^n$ points in those G-orbits of Ω on which N acts trivially.

(e) *The set $\pi = \{XG'/G' | X$ is a point-stabilizer not containing $N\} \cup \{N/G'\}$ is a partition of G/G'.*

By (d) we may assume that $k = p^n$. By Proposition 5(iii), G/G' acts on the set of all G'-orbits as a sharp irredundant group of type $\{p^n\}$ and so every point-stabilizer is a component of a partition. Since the point-stabilizers in the action of G/G' on G'-orbits are of the form XG'/G' where X is the stabilizer of a point of Ω not containing N, or N/G', the claim follows.

To conclude the proof of the theorem it is enough to observe that the number of conjugacy classes of the point-stabilizers not containing N equals the number of G-orbits on which N acts non-trivially. Since by (d) we may assume $k = p^n$, this number equals p^n and the claim follows. ∎

Proof of Theorem 2: Let G be a sharp irredundant non-abelian p-group, $p > 2$, of type $\{k\}$ on a set Ω. Let c be the class of G and let $|G : G'| = p^{2n}$. Let N be the normal component of π_Ω. Then by point (d) in the proof of Theorem 1 we may assume that $k = p^n$. Since $\gamma_c(G) \leq Z(G) \cap N$, from Lemma 6 it follows at once that $|\gamma_c(G)| \leq p^n$. To show the remaining part of claim (i) we prove first that

(∗) *if X is a point-stabilizer not containing N, then XG' acts on a subset of Ω as a sharp irredundant group of type $\{p^n\}$.*

Let Y be a point-stabilizer not containing N, $Y \neq X$, and suppose that $XG' \cap Y \neq 1$. Then $XG' \cap YG' > G'$ and, from point (e) in the proof of Theorem 1, it follows $XG' = YG'$, that is $Y \leq XG'$. Moreover, Y is a conjugate of X, since each conjugacy class of point-stabilizers of G determines a component of the partition $\pi = \{XG'/G' | X \text{ is a point-stabilizer not containing } N\} \cup \{N/G'\}$. Thus XG' acts regularly on all but one of the G-orbits of Ω on which N acts non-trivially. Namely, XG' acts non-regularly on that G-orbit of Ω whose point-stabilizers are conjugates of X. On that G-orbit, XG' acts with p^n orbits of length $|XG' : X| = |G : X|/p^n$. Since each G-orbit on which N acts trivially is also a non-regular XG'-orbit, we have that XG' has exactly $p^n + 1$ non-regular orbits. Hence it acts as a sharp irredundant group of type $\{p^n\}$ on the union of those orbits.

Let us now prove that

(∗∗) *for each $i = 2, \ldots, c-1$, $|\gamma_i(G) : \gamma_{i+1}(G)| = p^n$ and $\gamma_{i+1}(G) = \gamma_{i+1-j}(X\gamma_{j+1}(G))$ for every point-stabilizer X not containing N and for every positive integer $j < i$.*

Let us fix an i and suppose that we have already proved the claim for every $h < i$. In particular, we have that $\gamma_{h+1}(G) = \gamma_2(X\gamma_h(G))$ for every point-stabilizer X not containing N and for every positive integer $h < i$. Hence from (∗) it follows that $X\gamma_i(G)$ is a sharp irredundant group of type $\{p^n\}$ for each point-stabilizer X not containing N. Let L be a maximal subgroup of $\gamma_{i+1}(G)$ containing $\gamma_{i+2}(G)$. Since $\gamma_i(G)/L$ is not central in G and G is generated by the elements not contained in N, it follows that there exists a point-stabilizer Y not containing N such that $Y \not\leq C_G(\gamma_i(G)/L)$. Hence $Y\gamma_i(G)/L$ is non-abelian.

Since clearly $\gamma_2(Y\gamma_i(G)) \le \gamma_{i+1}(G)$, it must be $\gamma_2(Y\gamma_i(G)/L) = \gamma_{i+1}(G)/L$. Repeated applications of Proposition 5(iii) yield that $Y\gamma_i(G)/L$ is a sharp irredundant group of type $\{p^n\}$. Then, by Proposition 8, $Y\gamma_i(G)/L$ is an extra-special p-group, whence $p^{2n} = |Y\gamma_i(G)/L : \gamma_2(Y\gamma_i(G)/L)| = |Y\gamma_i(G) : \gamma_{i+1}(G)|$. By Theorem 1(ii) we have that $|Y\gamma_i(G) : \gamma_i(G)| = p^n$ and so $|\gamma_i(G) : \gamma_{i+1}(G)| = p^n$ as claimed. Now note that from Theorem 1(ii) and from a theorem by Khukhro [7, Theorem 1.6.1], it follows that no point-stabilizer not containing N centralizes $\gamma_i(G)/L$, since $|\gamma_i(G)/L| = p^{n+1}$. Therefore, every point-stabilizer X not containing N can be taken in the place of the subgroup Y above and so we have that $\gamma_{i+1}(G) = \gamma_2(X\gamma_i(G))$ for every point-stabilizer X not containing N. Since, by assumption, for $j < i$ we have $\gamma_i(G) = \gamma_{i-j}(X\gamma_{j+1}(G))$, it follows that $\gamma_{i+1}(G) = \gamma_2(X\gamma_i(G)) \le \gamma_{i-j+1}(X\gamma_{j+1}(G)) \le \gamma_{i+1}(G)$, whence $\gamma_{i+1}(G) = \gamma_{i-j+1}(X\gamma_{j+1}(G))$ as claimed.

In order to prove that the lower central series and the upper central series of G coincide, let L be a maximal subgroup of $\gamma_c(G)$ and let X be a point-stabilizer not containing N. By the same argument used to prove (**) we have that $X\gamma_c(G)/L$ is an extra-special p-group. It follows that $\gamma_c(G) = Z(G)$. Thus if $c = 2$ we are done. If $c > 2$, then by Proposition 5(iii), $G/Z(G)$ is a non-abelian sharp irredundant p-group of finite type and the claim follows immediately by induction.

In order to prove (iii), let H be a normal subgroup of G, $H \le G'$. By point (ii), $H \cap \gamma_c(G) = H \cap Z(G) \ne 1$. By Proposition 5(iii), $\overline{G} = G/H \cap \gamma_c(G)$ is a sharp irredundant p-group of finite type. Then by inductive hypothesis there exists an index i such that $\gamma_i(\overline{G}) \le \overline{H} \le \gamma_{i+1}(\overline{G})$. Hence $\gamma_i(G) \le H \le \gamma_{i+1}(G)$, as stated.

Point (iv) is an immediate consequence of (iii) and Proposition 5(iii). ∎

Proof of Theorem 3: Let G be a finite non-abelian p-group, $p > 2$, with a faithful representation as a sharp irredundant group of type $\{k\}$ on a set Ω. Let $|G : G'| = p^{2n}$. Let M be a maximal subgroup of G', normal in G. Then by Proposition 5(iii), G/M has a faithful representation as a sharp irredundant group of type $\{k\}$. So by Proposition 8 it is an extra-special p-group of order p^{2n+1}. Thus $k = p^n + |\pi| - 1$ for some partition π of an elementary abelian p-group of order p^n by Theorem 7 and the Remark after Theorem 7.

Conversely, let π be a partition of an elementary abelian p-group of order p^{2n}. Since Theorem 1 holds for G, the proof of the reverse implication of that theorem shows that G has a faithful representation as a sharp irredundant group of type $\{p^n\}$. Then the argument used in the proof of Theorem 7 to show that an extra-special p-group of exponent p and order p^{2n+1} has a faithful representation as a sharp irredundant group of type $\{p^n + |\pi| - 1\}$ applies here and we get that G has a faithful representation as sharp irredundant group of type $\{p^n + |\pi| - 1\}$. ∎

5. Examples

Example 1: For each odd prime p and positive integers m, n with $n < p$, there exists a metabelian p-group of exponent p, order q^{n+1}, where $q = p^m$, and class n, which has a faithful representation as a sharp irredundant group of type $\{q\}$.

Proof: Let p, m, n, q be as in the statement. Let $\mathbb{F}$ be a field of order q, $\mathbb{F}_p$ its prime subfield and set $V = \mathbb{F}^n$.

In the ring $M(n, q)$ of all square matrices of degree n with coefficients in $\mathbb{F}$, we denote by I the identity matrix and by J_i, $i = 2, \ldots, n$ the matrix with 1 on the $(i-1)$th lower diagonal and 0 elsewhere. Then $J_i^2 = J_{2i-1}$ for $i \leq (n+1)/2$ and $J_i^2 = 0$ for $i > (n+1)/2$. Let $a_1, \ldots, a_m$ be $\mathbb{F}_p$-linearly independent elements of $\mathbb{F}$ and define $A_i = I + a_i J_2 \in M(n, q)$, $i = 1, \ldots, m$, and $H = \langle A_1, \ldots, A_m \rangle \leq GL(n, q)$. For each $1 \leq i, j \leq m$ we have that $A_i A_j = A_j A_i$; moreover, A_i has order p. Hence H is an elementary abelian p-group of order $p^m = q$.

Consider the usual action of H on V by right multiplication of row vectors, which we denote, using the exponential notation, by $\mathbf{v}^B$ for each $\mathbf{v} \in V$ and $B \in H$. Define G to be the semidirect product $V \rtimes H$ with respect to this action.

It is easy to check that, for $k = 2, \ldots, n+1$,

$$\gamma_k(G) = \{(v_i, \ldots, v_n) \in V | v_i = 0 \text{ for } i = n-k+2, \ldots, n\}$$

and so G has nilpotency class n. Since $n < p$, G is a regular p-group and so, being generated by elements of order p, it has exponent p.

We claim that $C_V(g) = Z(G)$ for each $g \in G \setminus V$. Trivially, we have only to show that $C_V(g) \subseteq Z(G)$. Let $g \in G \setminus V$. Then $g = A_1^{\alpha_1} \cdots A_m^{\alpha_m} \mathbf{u}$, where $\mathbf{u} \in V$, $\alpha_i \in \mathbb{F}_p$ and they are not all zero. Now $A_1^{\alpha_1} \cdots A_m^{\alpha_m} = I + \sum_{i=2}^{n} b_i J_i$, where $b_i \in \mathbb{F}$ and $b_2 = \alpha_1 a_1 + \cdots + \alpha_m a_m$. Thus $b_2 \neq 0$, since the a_i are $\mathbb{F}_p$-linearly independent. If $\mathbf{v} = (v_1, \ldots, v_n) \in C_V(g)$, then we have

$$0 = [\mathbf{v}, g] = \left(\sum_{i=2}^{n} b_i v_i, \ldots, \sum_{i=2}^{n-k+1} b_i v_{i+k-1}, \ldots, b_2 v_n, 0 \right).$$

Therefore $\sum_{i=2}^{n-k+1} b_i v_{i+k-1} = 0$ for each $k = 1, \ldots, n-1$. Since $b_2 \neq 0$, it follows that $v_i = 0$ for each $i = 2, \ldots, n$, whence $\mathbf{v} \in Z(G)$ as claimed.

Let $\mathbf{w} = (0, 1, \ldots, 1) \in V$ and for each $x \in \mathbb{F}$, $i = 1, \ldots, m$, define $\mathbf{w}_x^i = a_1^{-1} a_i x \mathbf{w}$. Moreover, set $H_x = \langle A_1 \mathbf{w}_x^1, \ldots, A_m \mathbf{w}_x^m \rangle \leq G$. It is straightforward to verify that, for each $x \in \mathbb{F}$, H_x is an elementary abelian p-group of order q and $H_x \cap V = 1$. Moreover, we claim that, for every $x, y \in \mathbb{F}$, $g \in G$, condition $H_x \cap H_y^g \neq 1$ implies $x = y$ and $g \in N_G(H_x)$. To see this let $(A_1 \mathbf{w}_x^1)^{\alpha_1} \cdots (A_m \mathbf{w}_x^m)^{\alpha_m} = [(A_1 \mathbf{w}_y^1)^{\beta_1} \cdots (A_m \mathbf{w}_y^m)^{\beta_m}]^g$ be a non-trivial element in $H_x \cap H_y^g$, where, since $G = VH_y$, we may take $g \in V$. Then we have that

$$A_1^{\alpha_1} \cdots A_m^{\alpha_m} \equiv A_1^{\beta_1} \cdots A_m^{\beta_m} \mod V$$

whence $\alpha_i = \beta_i$ for each $i = 1, \ldots, m$. Thus we have that

$$(A_1 \mathbf{w}_x^1)^{\alpha_1} \cdots (A_m \mathbf{w}_x^m)^{\alpha_m} \equiv [(A_1 \mathbf{w}_y^1)^{\alpha_1} \cdots (A_m \mathbf{w}_y^m)^{\alpha_m}]^g \mod G'$$

whence we get the following relation, which we write in additive notation since it involves only elements of V:

$$\alpha_1 \mathbf{w}_x^1 + \cdots + \alpha_m \mathbf{w}_x^m \equiv \alpha_1 \mathbf{w}_y^1 + \cdots + \alpha_m \mathbf{w}_y^m \mod G'$$

and so

$$(\alpha_1 a_1^{-1} a_1 x + \cdots + \alpha_m a_1^{-1} a_m x)\mathbf{w} \equiv (\alpha_1 a_1^{-1} a_1 y + \cdots + \alpha_m a_1^{-1} a_m y)\mathbf{w} \mod G'.$$

Thus, since $\mathbf{w} \notin G'$, we have that $\alpha_1(x - y)a_1 + \cdots + \alpha_m(x - y)a_m = 0$ in $\mathbb{F}$. Since a_i are $\mathbb{F}_p$-linearly independent and α_i are not all zero, we get that $x = y$. Therefore g centralizes the element $(A_1 \mathbf{w}_x^1)^{\alpha_1} \cdots (A_m \mathbf{w}_x^m)^{\alpha_m} \notin V$ and so, by what we proved above, $g \in Z(G)$. Thus $g \in N_G(H_x) = Z(G)H_x$.

Now for each $x \in \mathbb{F}$, H_x has $|G : N_G(H_x)| = q^{n-1}$ conjugates. Therefore

$$\left| V \cup \bigcup_{x \in \mathbb{F}, g \in G} H_x^g \right| = q^n + qq^{n-1}(q - 1) = q^{n+1} = |G|.$$

Hence $\pi = \{V\} \cup \{H_x^g | x \in \mathbb{F}, g \in G\}$ is a normal partition of G such that $|N_G(X) : X| = q$ for each $X \in \pi$. Furthermore, its components split into $q + 1$ conjugacy classes. Therefore, by Theorem 1 in [4], G has a faithful representation as a sharp irredundant group of type $\{q\}$. ∎

Example 2: For each odd prime p and positive integers l, m, n, with l a divisor of m and $n < p$, there exists a metabelian p-group of exponent p, order $p^{m(n+1)}$ and class n, which has a faithful representation as a sharp irredundant group of type $\{\sum_{i=1}^{m/l} r^i\}$, where $r = p^l$.

Proof: Let G be the group of Example 1 and let us use the same notation. Regard G/V as a vector space of degree m/l over a field with $r = p^l$ elements $\mathbb{F}_r$ and let π be the partition of G/V consisting of all subspaces of dimension 1. Then $|\pi| = 1 + r + \cdots + r^{m/l-1}$. Therefore, by Theorem 3, G has a faithful representation as a sharp irredundant group of type $\{\sum_{i=1}^{m/l} r^i\}$. ∎

Using Proposition 5, other examples can be constructed as quotients of the groups given in the previous examples.

We apply now the results of Theorem 1 and Theorem 2 to determine all irredundant sharp non-abelian p-groups of order at most p^5, $p \geq 3$. For the classification of p-groups of maximal class and order at most p^5 we refer to [1].

Example 3: A non-abelian p-group G of order at most p^5, $p \geq 3$, can be faithfully represented as an irredundant sharp group of finite type if and only if it is one of the following:

(i) an extra-special p-group of order p^3 or p^5 and exponent p;

(ii) $G_1 = \langle a,b,c,d \mid [b,a] = c, [c,a] = d, [d,a] = [b,c] = [b,d] = [c,d] = 1, a^p = b^p = c^p = d^p = 1\rangle$, where $p > 3$;

(iii) $G_2 = \langle a,b,c,d \mid [b,a] = c, [c,a] = d, [d,a] = [b,c] = [b,d] = [c,d] = 1, b^3 = d^{-1}, a^3 = c^3 = d^3 = 1\rangle$;

(iv) $G_3 = \langle a,b,c,d,e \mid [b,a] = c, [c,a] = d, [d,a] = e, [b,c] = e, [b,d] = [b,e] = [c,d] = [c,e] = [d,e] = 1, a^p = b^p = c^p = d^p = e^p = 1\rangle$, where $p > 3$;

(v) $G_4 = \langle a,b,c,d,e \mid [b,a] = c, [c,a] = d, [d,a] = e, [b,c] = [b,d] = [b,e] = [c,d] = [c,e] = [d,e] = 1, a^3 = d^3 = e^3 = 1, b^3 = d^{-1}e, c^3 = e^{-1}\rangle$.

Proof: First of all we show that the groups in (i)–(v) have a faithful representation as irredundant sharp groups of finite type. In the case of extra-special p-groups of exponent p, it follows from Theorem 7. If G is one of the groups in (ii)–(v), then it has maximal class and so it satisfies the conditions of Theorem 1 provided that, for every element g outside $N = C_G(Z_2(G))$, g has order p and $C_N(g) = Z(G)$. In any case, an easy calculation shows that $C_N(g) = Z(G)$ for every element $g \notin N$. Moreover, if G is as in (ii) or (iv), then it is a regular p-group and, since it is generated by elements of order p, it has exponent p. If G is a 3-group as in (iii) or (v), then a acts on N as a splitting automorphism of order 3, that is $xx^a x^{a^2} = 1$ for each $x \in N$. Hence, by a well known fact about splitting automorphisms (see [4, p. 824]), every element outside N has order 3.

Let now G be an irredundant sharp non-abelian p-group of finite type and order at most p^5, $p \geq 3$. If $|G| = p^3$, then G is an extra-special p-group of exponent p.

Let $|G| = p^4$. Then by Theorem 1, $|G : G'| = p^2$ and by Theorem 2(ii), G has class 3. By a result by Blackburn [1, p. 88] any group of order p^4 and class 3 has the presentation

$$G = \langle a, b, c, d | [b, a] = c, [c, a] = d, [d, a] = [b, c] = [b, d] = [c, d] = 1, \\ a^p = d^\delta, b^p d^{\binom{p}{3}} = d^\gamma, c^p = d^p = 1\rangle$$

where δ and γ are suitable integers.

By Theorem 1(ii), the normal component N must coincide with $C_G(\gamma_2(G))$ and every element outside N has order p. If $p > 3$, then G is a regular p-group and, since it has a non-trivial partition, it must have exponent p. Hence we must have $\delta = \gamma = 0$. Thus G is the group G_1. If $p = 3$, then the presentation of G becomes

$$G = \langle a, b, c, d | [b, a] = c, [c, a] = d, [d, a] = [b, c] = [b, d] = [c, d] = 1, \\ a^3 = d^\delta, b^3 d = d^\gamma, c^3 = d^3 = 1\rangle.$$

Then $N = \langle b, c, d\rangle$ and $(ab)^3 = d^\gamma$. Since every element outside N must have order 3, we must have $\delta = \gamma = 0$. Thus G is the group G_2.

Let $|G| = p^5$. By Theorem 1, $|G : G'| = p^2$, or p^4. If $|G : G'| = p^4$, then $|G'| = p$ and so, by Proposition 8, G is the extra-special group of order p^5 and exponent p. So let us assume $|G : G'| = p^2$. Then by Theorem 2, G has class 4 and, by Theorem 1, the subgroup N coincides with $C_G(\gamma_3(G))$ and every element outside N has order p.

If $p > 3$, then by [1, p. 88], G has the presentation

$$G = \langle a, b, c, d, e | [b, a] = c, [c, a] = d, [d, a] = e, [b, c] = e, [b, d] = [b, e] = 1, \\ a^p = e^\delta, b^p c^{\binom{p}{2}} d^{\binom{p}{3}} e^{\binom{p}{4}} = e^\gamma, c^p d^{\binom{p}{2}} e^{\binom{p}{3}} = 1, d^p e^{\binom{p}{2}} = 1, d^p = 1\rangle$$

where γ, δ are suitable integers. Since $p > 3$, G is a regular p-group, and since it has a non-trivial partition it must have exponent p. Hence the previous presentation becomes

$$G = \langle a, b, c, d, e | [b, a] = c, [c, a] = d, [d, a] = e, [b, c] = e, [b, d] = [b, e] = 1, \\ a^p = b^p = c^p = d^p = e^p = 1\rangle,$$

that is G is the group G_3.

If $p = 3$, then by [1, p. 88], G has the presentation

$$G = \langle a, b, c, d, e | [b, a] = c, [c, a] = d, [d, a] = e, [b, c] = e^\beta, [b, d] = [b, e] = 1, \\ a^3 = e^\delta, b^3 c^3 d = e^\gamma, c^3 d^3 e = 1, d^3 e^3 = 1, d^3 = 1\rangle$$

where either $\beta = 1, \gamma = 0, \delta = 0, 1, 2$ or $\beta = \delta = 0, \gamma = 1$ or $\beta = \gamma = 0, \delta = 0, 1$.

In any case $N = \langle b, c, d, e\rangle$ and $(ab)^3 = e^{\beta+\gamma+\delta}$. Since a and ab must have order 3, it must be $\beta = \gamma = \delta = 0$. Hence G is the group G_4. ∎

Let us now determine all values of $k < p^5$ such that there exists an irredundant sharp non-abelian p-group of type $\{k\}$. According to Theorem 3 we need to know the cardinalities of all non-trivial partitions of elementary abelian p-groups of order at most p^4.

Let A be an elementary abelian p-group of order p^n, $n \geq 2$.

If $n = 2$, then the set of all subgroups of order p is the only non-trivial partition of A. It has cardinality $p + 1$.

Let $n = 3$. The cardinalities of non-trivial partitions of A are:
$p^2 + p + 1$, if the partition consists of subgroups of order p;
$p^2 + 1$, if the partition contains a subgroup of order p^2.

Let $n = 4$. The cardinalities of non-trivial partitions of A are:
$p^3 + p^2 + p + 1$, if the partition consists of subgroups of order p;
$p^3 + 1$, if the partition contains a subgroup of order p^3;
$p^3 + p^2 - (m-1)p + 1$, for $1 \leq m \leq p^2 + 1$, if the partition consists of m subgroups of order p^2 (which are one-dimensional subspaces of A regarded as a vector space of dimension 2 over the field with p^2 elements) and $(p+1)(1+p^2-m)$ subgroups of order p.

Hence possible values for k are: $p, p^2, p^2 + p, p^3, p^3 + p^2, p^3 + p^2 + p, p^4$ and $p^4 + p^3 + p^2 - (m-1)p$, for $1 \leq m \leq p^2 + 1$. Example 1 and Theorem 3 assure that for each such number k there is an irredundant sharp non-abelian p-group of type $\{k\}$.

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